the expression on the right in (A5) is finite, and so the integral in (3) converges in quadratic mean.


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# Determination of the Merit Factor of Legendre Sequences

TOM HØHOLDT AND HELGE E. JENSEN

***Abstract*—Golay has used the ergodicity postulate to calculate that the merit factor $F$ of a Legendre sequence offset by a fraction $f$ of its length has an asymptotic value given by $1/F=(2/3)-4|f|+8f^2$, $|f|\le 1/2$, which gives $F=6$ for $|f|=1/4$. Here this is proved without using the ergodicity postulate.**

## I. Introduction

The merit factor of a sequence of $N$ elements $x_j$, $0\le j\le N-1$ of value $+1$ or $-1$, is defined by

$$F=N^2\Bigg/\left(2\sum_{k=1}^{N-1}c_k^2\right) \tag{1.1}$$

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where

$$c_k=\sum_{j=0}^{N-k-1}x_jx_{j+k}. \tag{1.2}$$

The merit factor was defined by Golay [1] who conjectured [2] that $F\le 12.32$ for all binary sequences, with the exception of the Barker sequence of length 13 for which $F=14.08$. In [3] Golay used what he called the "ergodicity postulate" to establish the asymptotic merit factor of offset Legendre sequences.

Legendre sequences have length $p$ (an odd prime) and are defined by

$$x_0=1 \quad\text{and}\quad x_j=\begin{cases}1, & \text{if } j \text{ is a square } (\text{mod } p),\\ -1, & \text{if } j \text{ is a nonsquare } (\text{mod } p).\end{cases} \tag{1.3}$$

An "offset" sequence is one in which a fraction $f$ of its elements is chopped off at one end of the Legendre sequence and appended at the other. Golay arrived at the formula

$$1/F=(2/3)-4|f|+8f^2, \qquad |f|\le\frac{1}{2}, \tag{1.4}$$

which we prove to be correct without using the ergodicity postulate.

In Section II we present a method for calculating the merit factor of any odd-length sequence, and in Section III we use the method in the case of offset Legendre sequences.

Odlyzko has told us that Stephens announced an independent solution of the problem for Legendre sequences. More recently, Van Lint informed us that Heath-Brown and Birch have also announced a proof.

## II. Calculation of the Merit Factor for an Odd-Length Sequence

Let $(x_j)$, $j=0,1,\cdots,N-1$ be any $\pm 1$ sequence of odd length $N$. Let $Q(z)=x_0+x_1z+\cdots+x_{N-1}Z^{N-1}$ be the $z$ transform of the sequence. A straightforward calculation yields

$$|Q(e^{i\theta})|^2=N+c_1(e^{i\theta}+e^{-i\theta})+\cdots+c_{N-1}(e^{i(N-1)\theta}+e^{-i(N-1)\theta}), \tag{2.1}$$

and therefore by putting $\epsilon_j=e^{(2\pi i/N)\cdot j}$,

$$\sum_{j=0}^{N-1}|Q(\epsilon_j)|^4=N^3+2N\sum_{k=1}^{N-1}c_kc_{N-k}+2N\sum_{k=1}^{N-1}c_k^2$$

and

$$\sum_{j=0}^{N-1}|Q(-\epsilon_j)|^4=N^3+2N(-1)^N\sum_{k=1}^{N-1}c_kc_{N-k}+2N\sum_{k=1}^{N-1}c_k^2.$$

Since $N$ is odd, we get

$$2\sum_{k=1}^{N-1}c_k^2=\frac{1}{2N}\left(\sum_{j=0}^{N-1}|Q(\epsilon_j)|^4+\sum_{j=0}^{N-1}|Q(-\epsilon_j)|^4\right)-N^2. \tag{2.2}$$

We put $S=\sum_{j=0}^{N-1}(|Q(\epsilon_j)|^4+|Q(-\epsilon_j)|^4)$ and obtain

$$1/F=S/2N^3-1. \tag{2.3}$$

It now follows from a well-known interpolation formula [4, p. 89] that

$$Q(z)=\frac{1}{N}\sum_{j=0}^{N-1}\epsilon_j\frac{z^N-1}{z-\epsilon_j}\cdot Q(\epsilon_j), \tag{2.4}$$

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and therefore,

$$Q(-\epsilon_k)=\frac{2}{N}\sum_{j=0}^{N-1}\frac{\epsilon_j}{\epsilon_k+\epsilon_j}Q(\epsilon_j) \tag{2.5}$$

so that

$$S_1=\sum_{k=0}^{N-1}|Q(-\epsilon_k)|^4=\frac{16}{N^4}\sum_{k=0}^{N-1}\left|\sum_{j=0}^{N-1}\frac{\epsilon_j}{\epsilon_j+\epsilon_k}Q(\epsilon_j)\right|^4 \tag{2.6}$$

$$=\frac{16}{N^4}\sum_{\substack{a,b=0\\c,d}}^{N-1}Q(\epsilon_a)\bar{Q}(\epsilon_b)Q(\epsilon_c)\bar{Q}(\epsilon_d)\epsilon_a\cdot\epsilon_c$$

$$\cdot\sum_{j=0}^{N-1}\frac{1}{\epsilon_a+\epsilon_j}\cdot\frac{\epsilon_j}{\epsilon_b+\epsilon_j}\cdot\frac{1}{\epsilon_c+\epsilon_j}\cdot\frac{\epsilon_j}{\epsilon_d+\epsilon_j}. \tag{2.7}$$

The inner sum in (2.7) can be calculated. The result depends, of course, on the relations between $a$, $b$, $c$, and $d$. In all cases we use partial fraction expansion of

$$\frac{z^2}{(z+\epsilon_a)(z+\epsilon_b)(z+\epsilon_c)(z+\epsilon_d)},$$

which corresponds to the term in the inner sum.

*Case 1:* Note that $a$, $b$, $c$, and $d$ are mutually different:

$$\sum_{j=0}^{N-1}\frac{1}{\epsilon_a+\epsilon_j}\cdot\frac{\epsilon_j}{\epsilon_b+\epsilon_j}\cdot\frac{1}{\epsilon_c+\epsilon_j}\cdot\frac{\epsilon_j}{\epsilon_d+\epsilon_j}$$

$$=\sum_{j=0}^{N-1}\frac{\alpha_1}{\epsilon_a+\epsilon_j}+\sum_{j=0}^{N-1}\frac{\alpha_2}{\epsilon_b+\epsilon_j}+\sum_{j=0}^{N-1}\frac{\alpha_3}{\epsilon_c+\epsilon_j}+\sum_{j=0}^{N-1}\frac{\alpha_4}{\epsilon_d+\epsilon_j}$$

$$=\left(\frac{\alpha_1}{\epsilon_a}+\frac{\alpha_2}{\epsilon_b}+\frac{\alpha_3}{\epsilon_c}+\frac{\alpha_4}{\epsilon_d}\right)\sum_{l=0}^{N-1}\frac{1}{1+\epsilon_l},$$

but here the term in the brackets is zero, so the whole sum has value zero.

*Case 2:* We have $a=c$, and $a$, $b$, and $d$ are mutually different. We get

$$\sum_{j=0}^{N-1}\frac{\alpha_1}{\epsilon_a+\epsilon_j}+\sum_{j=0}^{N-1}\frac{\alpha_2}{\epsilon_b+\epsilon_j}+\sum_{j=0}^{N-1}\frac{\alpha_3}{\epsilon_d+\epsilon_j}+\sum_{j=0}^{N-1}\frac{\alpha_4}{(\epsilon_a+\epsilon_j)^2}$$

$$=\left(\frac{\alpha_1}{\epsilon_a}+\frac{\alpha_2}{\epsilon_b}+\frac{\alpha_3}{\epsilon_d}+\frac{\alpha_4}{\epsilon_a^2}\right)\sum_{l=0}^{N-1}\frac{1}{1+\epsilon_l}$$

$$+\frac{\alpha_4}{\epsilon_a^2}\sum_{l=0}^{N-1}\left[\frac{1}{(1+\epsilon_l)^2}-\frac{1}{(1+\epsilon_l)}\right]$$

$$=\frac{\alpha_4}{\epsilon_a^2}\sum_{l=0}^{N-1}\left[\frac{1}{(1+\epsilon_l)^2}-\frac{1}{(1+\epsilon_l)}\right],$$

and here $\alpha_4=\epsilon_a^2/(\epsilon_b-\epsilon_a)(\epsilon_d-\epsilon_a)$ and

$$\sum_{l=0}^{N-1}\left[\frac{1}{(1+\epsilon_l)^2}-\frac{1}{1+\epsilon_l}\right]=-\sum_{l=0}^{N-1}\frac{1}{|1+\epsilon_l|^2}=-\frac{N^2}{4}$$

(see [5, p. 105]). Therefore, the sum in this case is

$$-\frac{N^2}{4}\cdot\frac{1}{(\epsilon_b-\epsilon_a)}\cdot\frac{1}{(\epsilon_d-\epsilon_a)}.$$

*Case 3:* Note that $a=b=c\neq d$. We get

$$\sum_{j=0}^{N-1}\frac{\alpha_1}{\epsilon_a+\epsilon_j}+\sum_{j=0}^{N-1}\frac{\alpha_2}{\epsilon_d+\epsilon_j}+\sum_{j=0}^{N-1}\frac{\alpha_3}{(\epsilon_a+\epsilon_j)^2}+\sum_{j=0}^{N-1}\frac{\alpha_4}{(\epsilon_a+\epsilon_j)^3}$$

$$=\left(\frac{\alpha_1}{\epsilon_a}+\frac{\alpha_2}{\epsilon_d}+\frac{\alpha_3}{\epsilon_a^2}+\frac{\alpha_4}{\epsilon_a^3}\right)\sum_{l=0}^{N-1}\frac{1}{1+\epsilon_l}$$

$$+\frac{\alpha_3}{\epsilon_a^2}\sum_{l=0}^{N-1}\left[\frac{1}{(1-\epsilon_l)^2}-\frac{1}{1+\epsilon_l}\right]$$

$$+\frac{\alpha_4}{\epsilon_a^3}\sum_{l=0}^{N-1}\left[\frac{1}{(1+\epsilon_l)^3}-\frac{1}{(1+\epsilon_l)}\right]$$

$$=0-\frac{\alpha_3}{\epsilon_a^2}\cdot\frac{N^2}{4}+\frac{\alpha_4}{\epsilon_a^3}\cdot\left(-\frac{3}{8}N^2\right)$$

where

$$\alpha_3=\frac{-\epsilon_a(2\epsilon_d-\epsilon_a)}{(\epsilon_d-\epsilon_a)^2}\quad\text{and}\quad\alpha_4=\frac{\epsilon_a^2}{\epsilon_d-\epsilon_a}.$$

Here

$$\sum_{l=0}^{N-1}\left[\frac{1}{(1+\epsilon_l)^3}-\frac{1}{(1+\epsilon_l)}\right]$$

is calculated by using the fact that the $1/(1+\epsilon_l)$ are the roots of $P(z)=z^N((z^{-1}-1)^N-1)$, and hence the $1/(1+\epsilon_\alpha)^3$ are the roots of $P_1(z)$, where

$$P_1(u^3)=P(u)P(\epsilon u)P(\epsilon^2u)\quad\text{and}\quad\epsilon=-\frac{1}{2}+i\frac{\sqrt{3}}{2}.$$

In Case 3, therefore, we obtain that the sum is

$$\frac{N^2}{8}\left(\frac{2(2\epsilon_d-\epsilon_a)}{\epsilon_a(\epsilon_d-\epsilon_a)^2}-\frac{3}{\epsilon_a(\epsilon_d-\epsilon_a)}\right).$$

*Case 4:* We have $a=b\neq c=d$; we get

$$\sum_{j=0}^{N-1}\frac{\alpha_1}{\epsilon_a+\epsilon_j}+\sum_{j=0}^{N-1}\frac{\alpha_2}{(\epsilon_a+\epsilon_j)^2}+\sum_{j=0}^{N-1}\frac{\alpha_3}{(\epsilon_c+\epsilon_j)}+\sum_{j=0}^{N-1}\frac{\alpha_4}{(\epsilon_c+\epsilon_j)^2}$$

$$=\left(\frac{\alpha_1}{\epsilon_a}+\frac{\alpha_2}{\epsilon_a^2}+\frac{\alpha_3}{\epsilon_c}+\frac{\alpha_4}{\epsilon_c^2}\right)\sum_{l=0}^{N-1}\frac{1}{1+\epsilon_l}$$

$$+\left(\frac{\alpha_2}{\epsilon_a^2}+\frac{\alpha_4}{\epsilon_c^2}\right)\sum_{l=0}^{N-1}\left[-\frac{1}{(1+\epsilon_l)^2}-\frac{1}{1+\epsilon_l}\right]$$

$$=0+\left(\frac{\alpha_2}{\epsilon_a^2}+\frac{\alpha_4}{\epsilon_c^2}\right)\left(-\frac{N^2}{4}\right)$$

where $\alpha_2=\epsilon_a^2/(\epsilon_a-\epsilon_c)^2$ and $\alpha_4=\epsilon_c^2/(\epsilon_a-\epsilon_c)^2$, so the sum in this case is

$$-\frac{N^2}{2}\left(\frac{1}{(\epsilon_a-\epsilon_c)^2}\right).$$

*Case 5:* We have $a=b=c=d$. Here we consider

$$\sum_{j=0}^{N-1}\frac{\epsilon_a^2\cdot\epsilon_j^2}{(\epsilon_j+\epsilon_a)^4}=\sum_{l=0}^{N-1}\frac{1}{|1+\epsilon_l|^4}=\frac{1}{16}\left(\frac{1}{3}N^4+\frac{2}{3}N^2\right),$$

which follows easily from [6, p. 223]. Note that we here include $\epsilon_a\cdot\epsilon_c$ from (2.7) in the inner sum.

Using these results, we obtain

$$S_1 = \frac{16}{N^4}(A+B+C+D)$$

where

$$A = \frac{1}{16}\left(\frac{1}{3}N^4 + \frac{2}{3}N^2\right)\sum_{a=0}^{N-1}|Q(\epsilon_a)|^4$$

$$B = \frac{N^2}{8}\sum_{\substack{a,b=0\\a\neq b}}^{N-1} 2|Q(\epsilon_a)|^2\left(\overline{Q}(\epsilon_a)Q(\epsilon_b)\epsilon_b + Q(\epsilon_a)\overline{Q}(\epsilon_b)\epsilon_a\right)\left(\frac{\epsilon_a+\epsilon_b}{(\epsilon_b-\epsilon_a)^2}\right)$$

$$C = -\frac{N^2}{4}\sum_{\substack{a,b,c=0\\a\neq b\neq c}}^{N-1} 2|Q(\epsilon_a)|^2\left(\frac{Q(\epsilon_b)\overline{Q}(\epsilon_c)\epsilon_a\epsilon_b + \overline{Q}(\epsilon_b)Q(\epsilon_c)\epsilon_a\epsilon_c}{(\epsilon_b-\epsilon_a)(\epsilon_c-\epsilon_a)}\right)$$

$$-\frac{N^2}{4}\sum_{\substack{a,b,c=0\\a\neq b\neq c}}^{N-1}\frac{Q^2(\epsilon_a)\overline{Q}(\epsilon_b)\overline{Q}(\epsilon_c)\epsilon_a^2 + \overline{Q}^2(\epsilon_a)Q(\epsilon_b)Q(\epsilon_c)\epsilon_b\epsilon_c}{(\epsilon_b-\epsilon_a)(\epsilon_c-\epsilon_a)}$$

$$D = -\frac{N^2}{2}\cdot\frac{1}{2}\sum_{\substack{a,b=0\\a\neq b}}^{N-1}\frac{4|Q(\epsilon_a)|^2|Q(\epsilon_b)|^2\epsilon_a\epsilon_b + Q^2(\epsilon_b)\overline{Q}^2(\epsilon_a)\epsilon_b^2 + Q^2(\epsilon_a)\overline{Q}^2(\epsilon_b)\epsilon_a^2}{(\epsilon_a-\epsilon_b)^2}$$

and

$$\sum_{\nu=1}^{N-1}\frac{1}{|\epsilon_\nu-1|^2} = \frac{1}{12}N^2 - \frac{1}{12}. \tag{3.7}$$

## III. Offset Legendre Sequences

We now treat Legendre sequences as defined by (1.3). Long ago Gauss [7] knew that

$$Q(1)=1,\ Q(\epsilon_j) = \begin{cases} 1+x_j\sqrt{N}, & \text{if } N\equiv 1\ (\text{mod}\,4),\\ 1+ix_j\sqrt{N}, & \text{if } N\equiv 3\ (\text{mod}\,4),\end{cases}\quad j\neq 0. \tag{3.1}$$

Let $Q_t(z)$ denote the $z$ transform of an offset Legendre sequence arising from $t$ cyclic left shifts of a Legendre sequence. We then have

$$Q_t(\epsilon_j) = \epsilon_j^{-t}Q(\epsilon_j). \tag{3.2}$$

Therefore,

$$|Q_t(1)| = 1 \tag{3.3}$$

and

$$|Q_t(\epsilon_j)|^2 = \begin{cases} 1+N+2x_j\sqrt{N}, & \text{if } N\equiv 1\ (\text{mod}\,4),\\ 1+N, & \text{if } N\equiv 3\ (\text{mod}\,4),\end{cases}\quad j\neq 0, \tag{3.4}$$

which yields

$$|Q_t(\epsilon_j)|^4 = \begin{cases} N^2+4x_jN^{3/2}+6N+4x_j\sqrt{N}+1, \\ \qquad\qquad \text{if } N=1\ (\text{mod}\,4),\\ N^2+2N+1,\\ \qquad\qquad \text{if } N=3\ (\text{mod}\,4),\end{cases}\quad j\neq 0. \tag{3.5}$$

We are only interested in asymptotic results, so in the following calculations we neglect all but the highest (in $N$) order term of $Q_t(\epsilon_j)$, $|Q_t(\epsilon_j)|^2$, and $|Q_t(\epsilon_j)|^4$.

This is allowed by crude estimations where we only consider the numerical value and use

$$\sum_{\nu=1}^{N-1}\frac{1}{|\epsilon_\nu-1|} \leq N\log N \tag{3.6}$$

Here (3.6) follows from

$$\sum_{\nu=1}^{N-1}\frac{1}{|\epsilon_\nu-1|} = \frac{1}{2}\sum_{\nu=1}^{N-1}\frac{1}{\left|\sin\frac{\nu\pi}{N}\right|} = \sum_{\nu=1}^{(N-1)/2}\frac{1}{\left|\sin\frac{\nu\pi}{N}\right|}.$$

Hence if we use that $2/\pi x \leq \sin x \leq x$ for $x\in[0,\pi/2]$ and the logarithmic bound on the partial sum of the harmonic series, we get the result. Also, note from [6, p. 224] that $(1/12)N^2-(1/12)$ of (3.7) may be replaced by $(1/4)\sum_{\nu=1}^{N-1}1/(\sin^2(\pi\nu/N))$.

In the sequel we use $\sim$ to indicate that lower order terms are neglected. We now treat each of the expressions $A$, $B$, $C$, and $D$ singly:

$$A \sim \frac{1}{16}\left(\frac{1}{3}N^4\cdot N^3\right) \qquad \text{by (3.5)}$$

$$B \sim \frac{N^2}{8}\cdot 2$$

$$\cdot N\sum_{\substack{a,b=0\\a\neq b}}^{N-1}\frac{\left(\epsilon_b\cdot\epsilon_b^{-t}\epsilon_a^t x_a x_b N + \epsilon_a\cdot\epsilon_a^{-t}\cdot\epsilon_b^t x_a x_b N\right)(\epsilon_a+\epsilon_b)}{(\epsilon_b-\epsilon_a)^2}$$

$$= \frac{N^4}{4}\sum_{\substack{a,b=0\\a\neq b}}^{N-1}\frac{\epsilon_{a-b}^{t+1}+\epsilon_{a-b}^{2-t}+\epsilon_{a-b}^{t}+\epsilon_{a-b}^{1-t}}{(1-\epsilon_{a-b})^2}x_a x_b$$

$$= \frac{N^4}{4}\sum_{k=1}^{N-1}(c_k+c_{N-k})\frac{\epsilon_k^{t+1}+\epsilon_k^{2-t}+\epsilon_k^{t}+\epsilon_k^{1-t}}{(1-\epsilon_k)^2}$$

$$= \frac{N^4}{4}\cdot O(N^2) \tag{3.8}$$

where we use (3.7) and the fact (e.g., [3]) that $|c_k + c_{N-k}| \leq 3$ for Legendre sequences.

The sum $C$ is again $O(N^6)$. This can be seen by summing first with respect to $a$, using $\sum_{k=1}^{N-1}1/(1-\epsilon_k) = (N-1)/2$, and then using a rewriting similar to the one that gave the two last expressions in $B$. To treat $D$, we have

$$D = \frac{3N^2}{4}\sum_{\substack{a,b=0\\ a\neq b}}^{N-1}\frac{|Q_t(\epsilon_a)|^2|Q_t(\epsilon_b)|^2}{|1-\epsilon_{a-b}|^2}$$
$$+\frac{N^2}{4}\sum_{\substack{a,b=0\\ a\neq b}}^{N-1}\frac{|Q_t(\epsilon_a)|^2|Q_t(\epsilon_b)|^2+2\mathrm{Re}Q_t^2(\epsilon_b)\overline{Q}_t^2(\epsilon_a)\cdot\epsilon_{b-a}}{|1-\epsilon_{a-b}|^2}$$
$$\sim\frac{3N^2}{4}\cdot N^3\sum_{k=1}^{N-1}\frac{1}{|1-\epsilon_k|^2}+\frac{N^2}{4}\cdot N^3\sum_{k=1}^{N-1}\frac{1+\epsilon_k^{2t-1}+\epsilon_{-k}^{2t-1}}{|1-\epsilon_k|^2}$$
$$=\frac{3N^5}{4}\left(\frac{1}{12}N^2+O(N)\right)$$
$$+\frac{N^5}{4}\left(\frac{1}{4}N^2-\frac{1}{4}+(2t-1)(2t-2)-(2t-1)(N-1)\right)$$

since the last sums equals

$$\sum_{k=1}^{N-1}\left[\frac{3}{|1-\epsilon_k|^2}-\sum_{j=1}^{2t-1}j\left(\epsilon_k^{2t-j-1}+\epsilon_{-k}^{2t-j-1}\right)+2t-1\right]$$
$$=\frac{1}{4}N^2-\frac{1}{4}+2\sum_{j=1}^{2t-2}j-2(2t-1)(N-1)+(2t-1)(N-1)$$
$$=\frac{1}{4}N^2-\frac{1}{4}+(2t-1)(2t-2)-(2t-1)(N-1).$$

We can now summarize. By (2.3), we have

$$1/F=\frac{1}{2N^3}(S)-1$$
$$=\frac{1}{2N^3}\left(N^3+S_1\right)-1$$
$$=\frac{1}{2N^3}\left(N^3+\frac{16}{N^4}\left[\frac{1}{16}\left(\frac{1}{3}N^7\right)+\frac{1}{16}N^7\right.\right.$$
$$\left.\left.+\frac{N^5}{4}\left(\frac{1}{4}N^2+(2t-1)(2t-2)-(2t-1)(N-1)\right)\right]\right)-1$$
$$=\frac{2}{3}+8\left(\frac{t}{N}\right)^2-4\left(\frac{t}{N}\right)$$
$$=\frac{2}{3}+8f^2-4|f|$$

where $|f|=t/N$ in agreement with Golay's result.

## IV. Conclusion

The formula (2.2) that was used in our calculations grew out of an attempt to calculate the merit factor using integrals. If, as before, $Q(z)$ is the $z$ transform of the sequence $x_0, x_1, \cdots, x_{N-1}$, simple calculations give

$$\frac{1}{2\pi}\int_0^{2\pi}|Q(e^{i\theta})|^4\,d\theta=N^2+2\sum_{k=1}^{N-1}c_k^2. \tag{4.1}$$

This also follows from (2.1) by means of Parseval's formula.

Polynomials with $\pm 1$ coefficients considered on the unit circle in the complex plane have been intensively studied by Littlewood and many others. Unfortunately, there have been no results on integrals of the type (4.1), which can give new information on the behavior of the merit factor. (The situation will probably turn out to be the reverse.) However, one can try to evaluate the integral using interpolation, and the result [8, theorem 2.6] then gives (2.2) for odd $N$. This identity can also be verified by direct calculation (easy—when you know the result). The method then depends on our knowledge of the values of $Q(z)$ for $z=\exp(i2\pi k/N)$, $k=0,1,\cdots,N-1$. For Legendre sequences these values are known and simple to deal with, but in other situations it should be possible to obtain new information using this method.

The method used by Golay in [3] is quite different. He assumes —and this is the ergodicity postulate—that if you are only interested in asymptotic results, it is allowed to treat the correlations as independent random variables. With this assumption you can then use the machinery of probability theory. For Legendre sequences, offset or not, this turns out to give the correct result. In [2] the same method leads to the general conclusion that the maximal value of the merit factor tends to $12.32\cdots$ when the length tends to infinity. We doubt that this is true and, therefore, also doubt that the ergodicity postulate can be used in general.

Concerning the terms $A, B, C, D$ at the end of Section II, we find it probable that, for sequences with good merit factor, the contributions from $B$ and $C$ will vanish asymptotically. Hence, the merit factor probably never exceeds the value 6.

This belief is supported, or at least is not rejected, by the result in [9]. Here the merit factor is calculated for several good sequences with lengths greater than 100. In every case the result is either strictly smaller than or suspiciously close to 6. We therefore make a new conjecture concerning the merit factor problem, namely, that asymptotically the maximum value of the merit factor is 6 and hence that offset Legendre sequences are optimal.

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# Bounds for the Size of Radar Arrays

A. BLOKHUIS AND H. J. TIERSMA

***Abstract*—Improved upper and lower bounds for the size of radar arrays are presented.**

## Introduction

From [1] we recall the following definitions.

*Definition 1:* A radar array is an $N\times M$ matrix of ones and zeros, with a single one per column, such that the horizontal

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